Mean first-passage time in the presence of telegraph noise and the Ornstein-Uhlenbeck process

M. Kuś* and K. Wódkiewicz†
Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131
(Received 23 November 1992)

We consider the problem of the escape time (mean first-passage time) from a given interval in the case when the noise is a sum of many independent random telegraph signals. We reduce the problem to the solution of a linear system of algebraic equations valid for arbitrary intensities and correlation times of the noise. The solution allows an easy investigation of the limiting case of the Ornstein-Uhlenbeck process. We find exact scaling laws obeyed by the mean first-passage times in the case of random telegraph signals and the Ornstein-Uhlenbeck process.

PACS numbers: 02.50.−r, 05.40.+j

I. INTRODUCTION

The problem of the mean first-passage time (MFPT) for a one-dimensional dynamical equation involving stochastic driving by an external process with correlation time has attracted recently attention of several groups of investigators [1–14]. Usually, one considers the situation when the dynamics of a system is governed by the stochastic differential equation:

$$\frac{d\rho}{dt} = F(\rho) + x(t)G(\rho), \tag{1.1}$$

where the coordinate ρ describes the state of the system and $x(t)$ is a stochastic process with zero mean and exponentially vanishing time correlation:

$$\langle x(t)\rangle = 0, \quad \langle x(t)x(t')\rangle = \frac{\Gamma}{\tau}\exp\left(-\frac{|t-t'|}{\tau}\right). \tag{1.2}$$

In the limit $\tau \to 0$ the process $x(t)$ approaches the white-noise limit:

$$\langle x(t)\rangle = 0, \quad \langle x(t)x(0)\rangle = 2\Gamma\delta(t-t'). \tag{1.3}$$

Further assumptions about the process $x(t)$ are also important: one can consider the dichotomous noise [2, 3, 5, 11], but if we demand that $x(t)$ is Markovian and Gaussian we are left with the single possibility [15] of the Ornstein-Uhlenbeck process [16]. In applications we have usually $G(\rho) = 1$, whereas $F(\rho)$ is some nonlinear potential. Equation (1.2) describes thus the overdamped motion in the nonlinear potential $F(\rho)$ under the influence of the stochastic driving noise $x(t)$.

The problem consists of calculation of the mean time after which the system coordinate ρ crosses for the first time the boundary of the prescribed region in which the system was initially put (e.g., the first time after which a particle escapes from the potential well). The evolution of the system, due to the finite value of the correlation time τ, is non-Markovian, which precludes the construction of the exact Fokker-Planck-type equation.

Various approximations, sometimes with seemingly conflicting results, and valid for different correlation times and intensities of the noise were developed (cf. [9] and [11] for critical discussion of various results). Various types of numerical simulations were performed [11–14]. We recommend the reference list in Ref. [6] for further bibliographical information.

In our two previous publications [17, 18], to which the present paper is a sequel, we developed a systematic procedure allowing for finding MFPT using the approximation of the Ornstein-Uhlenbeck process by a finite number of discrete independent jumping processes of the telegraph type. The resulting addition of a finite number of random telegraph signals (RTS) is usually called the pre-Gaussian noise [19], whereas in the limiting case of (appropriately scaled) RTS we recover the Ornstein-Uhlenbeck process.

In previous work [17, 18], we were able to derive a system of differential equations fulfilled by various MFPT's in the presence of pre-Gaussian noise. The equations had to be supplemented with appropriate boundary conditions. We have formulated these non-Markovian boundary conditions guided by the earlier results of Refs. [2, 4, 5] (see also [3]).

From the point of view of possible applications, the most interesting situation arises when $F(\rho)$ corresponds to a bistable potential. Postponing the general situation of arbitrary potentials $F(\rho)$ and $G(\rho)$ to a further publication, in the present paper we shall concentrate on the simplest situation in which $F(\rho) = 0$ and $G(\rho) = 1$:

$$\frac{d\rho}{dt} = x(t), \tag{1.4}$$

i.e., a particle driven by a purely stochastic force. The resulting $\rho(t)$ corresponds to a colored Wiener-Lévy stochastic process. In order to study the escape time for such a stochastic process we shall assume that initially the particle corresponding to $\rho(t)$ is put inside of some arbitrary interval $[A, B]$. The mean first-passage time (or escape time) is determined as the mean first time at which the particle reaches one of the ends of the interval. The same problem in the case of a dichotomous (two state) noise was considered in Ref. [5] and investigated numerically in Ref. [12]. In Ref. [7] the authors derived a general method of treating multivalued noise in the context of the mean first-passage time and applied it to rederive the mean first-passage distribution in the case of a non-Markovian dichotomous noise as well as to find the mean first-passage time in the physically interesting model with dichotomous noise in which the motion can proceed in only one direction (Giddings-Eyring model).

In the next section we formulate the problem in terms of the RTS. Sections III and IV are devoted to the analytical solutions of the problem, whereas Sec. V provides the numerical investigations of the limiting case of the Ornstein-Uhlenbeck process.

II. EQUATIONS FOR THE MEAN FIRST-PASSAGE TIME IN THE CASE OF RTS

In the case of the pre-Gaussian noise, the driving stochastic process $x(t)$ is a sum of a finite number of N independent Markov random telegraph signals $x_i(t), i = 1,\dots,N$ with the mutual correlation functions (see, e.g., Ref. [19] and references cited therein):

$$\langle x_i(t)x_j(t')\rangle = a^2 \exp\left(-\frac{|t-t'|}{\tau}\right). \tag{2.1}$$

The composed process,

$$x(t) = \sum_{i=1}^{N} x_i(t), \tag{2.2}$$

is thus characterized by the following autocorrelation:

$$\langle x(t)x(t')\rangle = \frac{\Gamma}{\tau}\exp\left(-\frac{|t-t'|}{\tau}\right), \tag{2.3}$$

where

$$\Gamma = Na^2\tau \tag{2.4}$$

and τ is the correlation time. To simplify the following considerations we assume that N is odd: $N = 2K - 1, K = 1, 2, \dots$.

It was shown in Ref. [18] that the MFPT for the dynamical variable ρ can be expressed as

$$T(\rho) = \sum_{n=-N}^{N} \left(\frac{1}{2}\right)^N \binom{N}{\frac{N+n}{2}} T_n(\rho). \tag{2.5}$$

The auxiliary functions $T_n(\rho)$, for $n = -N, -N + 2, \dots, N-2, N$ (having their own interpretation of mean first-passage time for some prescribed initial configurations of the telegraphs) fulfill a set of linear differential equations [18]:

$$\left(F(\rho)\frac{\partial}{\partial\rho} + naG(\rho)\frac{\partial}{\partial\rho} - \frac{N}{2\tau}\right)T_n(\rho) + \frac{1}{2\tau}\left(\frac{N-n}{2}T_{n+2}(\rho) + \frac{N+n}{2}T_{n-2}(\rho)\right) = -1, \tag{2.6}$$

with mixed boundary conditions:

$$\begin{aligned} T_n(A) &= 0 \quad \text{for } n < 0, \\ T_n(B) &= 0 \quad \text{for } n > 0. \end{aligned} \tag{2.7}$$

In our case, since $F(\rho) = 0$ and $G(\rho) = 1$, Eq. (2.6) takes a particularly simple form, which in matrix notation reads

$$\frac{d}{d\rho}\mathbf{T} = \frac{1}{2a\tau}\mathbf{M}\mathbf{T} + \frac{1}{a}\mathbf{w}, \tag{2.8}$$

where $\mathbf{T}$ is the column vector with $T_n(\rho), n = -N, -N + 2, \dots, N-2, N$ as its components. The $(N+1)\times(N+1)$ matrix $\mathbf{M}$ has elements

$$\begin{aligned} M_{kl} &= -\frac{N-k}{2k}\delta_{k,l-2} + \frac{N}{k}\delta_{k,l} - \frac{N+k}{2k}\delta_{k,l+2}, \\ k,l &= -N, -N+2, \dots, N-2, N+2 \end{aligned} \tag{2.9}$$

and the components of the vector $\mathbf{w}$ read

$$w_k = \frac{1}{k}, \quad k = -N, -N+2, \dots, N-2, N. \tag{2.10}$$

III. SPECTRUM AND EIGENVECTORS OF M

In this section it will be more convenient to use the standard numbering of the components in Eq. (2.8), in which the indices run from 1 to $N + 1$. Using this convention we obtain for the matrix elements of $\mathbf{M}$:

$$\begin{aligned} M_{\alpha\beta} = &-\frac{N-\alpha+1}{2(\alpha-1)-N}\delta_{\alpha,\beta-1} + \frac{N}{2(\alpha-1)-N}\delta_{\alpha,\beta} \\ &-\frac{\alpha-1}{2(\alpha-1)-N}\delta_{\alpha,\beta+1}, \\ &\alpha,\beta = 1,2,\dots,N+1. \end{aligned} \tag{3.1}$$

Standard methods of solving equations of the type (2.8) require finding the eigenvalues and eigenvectors of the matrix $\mathbf{M}$. To this end let us denote

$$y = -N\lambda - N, \quad \mu = 2\lambda \tag{3.2}$$

and observe that the characteristic equation for the matrix $\mathbf{M}$ can be written in the form

$$\det(\mathbf{M}-\lambda)=\det\begin{pmatrix} -1-\lambda & 1 & 0 & & & \\ \frac{1}{N-2} & -\frac{N}{N-2}-\lambda & \frac{N-1}{N-2} & & & \\ 0 & \frac{2}{N-4} & -\frac{N}{N-4}-\lambda & & & \\ & & & \cdots & & \\ & & & \cdots & & \\ & & & -\frac{N-1}{N-2} & \frac{N}{N-2}-\lambda & -\frac{1}{N-2} \\ & & & 0 & -1 & 1-\lambda \end{pmatrix}=0. \tag{3.3}$$

Multiplying the consecutive rows of the matrix $\mathbf{M}-\lambda$ by $N, N-2, N-4, \ldots, -N+2, -N$, respectively, we obtain

$$\frac{1}{N^2(N-2)^2\cdots 1^2}\det(\mathbf{M}-\lambda)=\det\begin{pmatrix} y & N & 0 & & & \\ 1 & y+\mu & N-1 & & & \\ 0 & 2 & y+2\mu & & & \\ & & & \cdots & & \\ & & & \cdots & & \\ & & & N-1 & y+(N-1)\mu & 1 \\ & & & 0 & N & y+N\mu \end{pmatrix}=0. \tag{3.4}$$

Let us denote

$$W_n(y,\mu)=\det\begin{pmatrix} y & n & 0 & & & \\ 1 & y+\mu & n-1 & & & \\ 0 & 2 & y+2\mu & & & \\ & & & \cdots & & \\ & & & \cdots & & \\ & & & n-1 & y+(n-1)\mu & 1 \\ & & & 0 & n & y+n\mu \end{pmatrix}, \tag{3.5}$$

then

$$\det(\mathbf{M}-\lambda)=0 \Longleftrightarrow W_N(y,\mu)=0. \tag{3.6}$$

We shall prove that $W_n(y,\mu)$ fulfills the following recurrence relation:

$$W_n(y,\mu)=(y+n\sigma)W_{n-1}(y+\mu-\sigma,\mu), \tag{3.7}$$

where

$$\sigma=\tfrac{1}{2}\left(\mu+\sqrt{\mu^2+4}\right), \tag{3.8}$$

and consequently

$$\sigma^2=\mu\sigma+1. \tag{3.9}$$

Indeed, adding to each column of W_n all consecutive columns to the right multiplied by $\sigma, \sigma^2, \sigma^3$, etc. and then subtracting from each row the preceding one multiplied by σ, we obtain using (3.9)

$$W_n(y,\mu)=\det\begin{pmatrix} y+n\mu & n & 0 & & & \\ 0 & y-\sigma+\mu & n-1 & & & \\ 0 & 1 & y-\sigma+2\mu & & & \\ & & & \cdots & & \\ & & & \cdots & & \\ & & & n-2 & y-\sigma+(n-1)\mu & 1 \\ & & & 0 & n-1 & y-\sigma+n\mu \end{pmatrix}$$

from which the relation (3.7) follows immediately. The recurrence equation (3.7) is easy to solve and gives

$$W_n(y,\mu)=\prod_{k=0}^{n}[y+k\mu+(n-2k)\sigma]. \tag{3.10}$$

Taking into account our assumption that N is odd and the definitions (3.2) and (3.8) we obtain finally

$$\det(\mathbf{M}-\lambda)=0 \Longleftrightarrow 0=W_N(y,\mu)=\prod_{k=0}^{\frac{N-1}{2}}\left[N^2-(N-2k)^2(\lambda^2+1)\right], \quad (3.11)$$

hence the eigenvalues of $\mathbf{M}$ are given by

$$\lambda_j=\pm\sqrt{\left(\frac{N}{2j-1}\right)^2-1}, \quad j=1,2,\ldots,\frac{N+1}{2}=K. \quad (3.12)$$

In what follows, we shall denote by λ_j, for $j=1,\ldots,K$ only the non-negative eigenvalues.

All eigenvalues except $\lambda_K=0$ are nondegenerate. The doubly degenerate $\lambda_K=0$ eigenvalue has the geometric multiplicity equal one (i.e., there exists only one eigenvector belonging to this eigenvalue). To show this fact it is enough to show that there exist two different nonzero eigenvectors $\tilde{\mathbf{v}}^{(K)}$ and $\tilde{\mathbf{u}}^{(K)}$ such that

$$\mathbf{M}\tilde{\mathbf{u}}^{(K)}=\tilde{\mathbf{v}}^{(K)}, \quad \mathbf{M}\tilde{\mathbf{v}}^{(K)}=\mathbf{0}. \quad (3.13)$$

It can be easily checked by inspection that the above equations are fulfilled by vectors with the components

$$\tilde{u}_\alpha^{(K)}=-\left(\frac{N}{2}-\alpha+1\right), \quad \tilde{v}_\alpha^{(K)}=1, \qquad \alpha=1,\ldots,N+1=2K. \quad (3.14)$$

To find eigenvectors corresponding to the nonzero eigenvalues we have to solve the equation

$$\mathbf{M}\mathbf{u}^{(j)}=\lambda^j\mathbf{v}^{(j)}, \quad (3.15)$$

which in the component notation reads

$$-(N-\alpha+1)v_{\alpha+1}^{(j)}+Nv_\alpha^{(j)}-(\alpha-1)v_{\alpha-1}^{(j)} =[2(\alpha-1)-N]\lambda_j v_\alpha^{(j)}, \quad (3.16)$$

or, after shifting the variable $\alpha\to\alpha+1$,

$$(\alpha-N)v_{\alpha+2}^{(j)}+[N-(2\alpha-N)\lambda_j]v_{\alpha+1}^{(j)}-\alpha v_\alpha^{(j)}=0. \quad (3.17)$$

In order to solve this second-order recurrence equation we represent the components v_α in the following integral form:

$$v_\alpha^{(j)}=\int_{t_1}^{t_2}t^{\alpha-1}f^{(j)}(t)dt, \quad (3.18)$$

where the function $f^{(j)}$ as well as a contour of integration together with its end points t_1 and t_2 are to be found. (We suppress in our notation the dependence of f on α to simplify the following formulas.) Substituting (3.18) into the recurrence relation (3.17) we get

$$\int_{t_1}^{t_2}t^{\alpha-1}f^{(j)}(t)[\alpha u(t)+w(t)]dt=0, \quad (3.19)$$

where

$$u(t)=t^2-2\lambda_j t-1, \quad w(t)=-Nt^2+N(\lambda_j+1)t. \quad (3.20)$$

Integrating (3.19) by parts, we obtain

$$\int_{t_1}^{t_2}t^{\alpha-1}\left[f^{(j)}(t)w(t)-t\frac{d}{dt}\left(f^{(j)}(t)u(t)\right)\right]dt +t^\alpha f^{(j)}(t)u(t)|_{t_1}^{t_2}=0. \quad (3.21)$$

If we choose $f^{(j)}(t), t_1$, and t_2 in such a way that

$$t\frac{d}{dt}[f^{(j)}(t)u(t)]-f^{(j)}(t)w(t)=0 \quad (3.22)$$

and

$$t^\alpha f^{(j)}(t)u(t)|_{t_1}^{t_2}=0, \quad (3.23)$$

then (3.18) will be a solution of the recurrence equation (3.17). Solution of (3.22) is straightforward:

$$f^{(j)}(t)=\frac{C}{u(t)}\exp\left(\int\frac{w(t)}{tu(t)}dt\right), \quad (3.24)$$

which, after a short calculation using (3.20), gives

$$f^{(j)}(t)=C(t-\rho_1)^{-(K-j+1)}(t-\rho_2)^{-(K+j)}, \quad (3.25)$$

where

$$\rho_{1,2}=\lambda_j\pm\sqrt{\lambda_j^2+1}, \quad (3.26)$$

and we used $N=2K-1$. Now it can be seen that the condition (3.22) is fulfilled by choosing $t_1=t_2=0$ and we can perform the integration in (3.18) along a complex path encircling one of the points ρ_1,ρ_2 (in fact, one can prove that any choice of a contour leads to an equivalent result). Taking the contour passing through $t=0$ and closing around the point ρ_1 and calculating the integral with the help of Cauchy theorem we get, up to a normalization constant,

$$v_\alpha^{(j)}=\rho_1^{\alpha-1}\sum_{l=0}(-1)^l\binom{K-j}{l}\binom{\alpha-1}{l} \times\binom{2K-1}{l}^{-1}\left(1-\frac{\rho_2}{\rho_1}\right)^l, \qquad \alpha=1,\ldots,N+1=2K, \quad (3.27)$$

which can be conveniently written in terms of a hypergeometric function:

$$v_\alpha^{(j)} = \eta_j^{\alpha-1} F(-\alpha+1, j-K; -2K+1; 1+1/\eta_j^2),$$
$$\alpha = 1, \ldots, N+1 = 2K; \quad j = 1, \ldots, \frac{N+1}{2} = K, \tag{3.28}$$

where

$$\eta_j = \left(\frac{2K-1}{2j-1}\right) + \sqrt{\left(\frac{2K-1}{2j-1}\right)^2 - 1}. \tag{3.29}$$

Formula (3.28) represents the components of the eigenvector corresponding to the positive eigenvalue [cf. Eq. (3.12)]. One proves analogously that for eigenvectors corresponding to the negative eigenvalue one has

$$u_\alpha^{(j)} = \tilde{C} \eta_j{}^{1-\alpha} F(-\alpha+1, j-K; -2K+1; 1+\eta_j^2),$$
$$\alpha = 1, \ldots, N+1 = 2K, \quad j = 1, \ldots, \frac{N+1}{2} = K, \tag{3.30}$$

with η_j given, as previously, by (3.29). This time we have written a (until now arbitrary) normalization constant $\tilde{C}$ explicitly because taking $\tilde{C} = (-1)^{K-j}\eta_j^{2j-1}$ and using well-known symmetry properties of the hypergeometric function [21] we can write

$$u_\alpha^{(j)} = v_{2K-\alpha+1}^{(j)}. \tag{3.31}$$

For further use we rewrite the eigenvectors (3.14), (3.28), (3.30), and (3.31) using the original indexing. To this end we have to substitute $k = -N + 2(\alpha - 1)$. We obtain, thus, for the eigenvectors belonging to the zero eigenvalue (3.14)

$$\tilde{v}_k^{(K)} = 1, \quad \tilde{u}_k^{(K)} = \frac{k}{2}, \quad k = -N, -N+2, \ldots, N-1, N, \tag{3.32}$$

for the eigenvectors belonging to the positive eigenvalues (3.28):

$$v_k^{(j)} = \eta_j^{\frac{N+k}{2}} F\left(-\tfrac{N+k}{2}, j - \tfrac{N+1}{2}; -N; 1+1/\eta_j^2\right),$$
$$k = -N, -N+2, \ldots, N-1, N,$$
$$j = 1, \ldots, \frac{N-1}{2} = K-1, \tag{3.33}$$

whereas for those corresponding to the negative eigenvalues

$$u_k^{(j)} = v_{-k}^{(j)}, \quad k = -N, -N+2, \ldots, N-1, N,$$
$$j = 1, \ldots, \frac{N-1}{2} = K-1. \tag{3.34}$$

IV. BOUNDARY CONDITIONS AND EXACT SOLUTIONS FOR ARBITRARY N

The full solution $\mathbf{T}(\rho)$ of the inhomogeneous system of equations (2.8) is a sum of the general solution $\mathbf{T}^{(\text{hom})}(\rho)$ of the homogeneous part and a particular solution $\mathbf{T}^{(\text{inh})}(\rho)$ of the full inhomogeneous system. According to the results of the previous section, the general solution $\mathbf{T}^{(\text{hom})}(\rho)$ of the homogeneous part reads [as remarked above we denote by λ_j the value corresponding to the positive sign in Eq. (3.12)]

$$\mathbf{T}^{(\text{hom})}(\rho) = \sum_{j=1}^{K-1}\left[C_j \mathbf{v}^{(j)} \exp\left(\tfrac{\lambda_j \rho}{2a\tau}\right) + D_j \mathbf{u}^{(j)} \exp\left(-\tfrac{\lambda_j \rho}{2a\tau}\right)\right] + C_K \tilde{\mathbf{v}}^{(K)} + D_K\left(\rho \tilde{\mathbf{v}}^{(K)} + 2a\tau \tilde{\mathbf{u}}^{(K)}\right). \tag{4.1}$$

The components of a particular solution $T^{(\text{inh})}$ of the full (inhomogeneous) equation(2.8) can be written as

$$T_n^{(\text{inh})}(\rho) = -\frac{1}{2Na^2\tau}\rho^2 - \frac{n}{N}\frac{\rho}{a} - \frac{n^2}{2N}\tau = -\frac{1}{2\Gamma}\left(\rho \tilde{v}_n^{(K)} + 2a\tau \tilde{u}_n^{(K)}\right)^2,$$
$$n = -N, -N+2, \ldots, N-1, N, \tag{4.2}$$

as can be easily checked by inspection using (2.8) and (2.9). Equations(4.1) and (4.2) give the general solution of the problem as a sum of the homogeneous (4.1) and inhomogeneous (4.2) parts and must be supplemented with the boundary conditions (2.7).

Without losing generality, we can assume that the interval in question is symmetric around $\rho = 0$, i.e., $A = -\rho_0 = -B$. From the boundary conditions (2.7) we infer immediately using (3.31),

$$D_K = 0, \quad C_j = D_j, \quad j = 1, \ldots, K-1, \tag{4.3}$$

and we can rewrite the full solution (4.1) and (4.2) in the form

$$T_n(\rho) = \sum_{j=1}^{K} C_j\left(v_n^{(j)} \exp(\tfrac{\lambda_j \rho}{2a\tau}) + u_n^{(j)} \exp(-\tfrac{\lambda_j \rho}{2a\tau})\right) - \frac{1}{2\Gamma}\left(\rho v_n^{(K)} + 2a\tau \tilde{u}_n^{(K)}\right)^2,$$
$$n = -N, -N+2, \ldots, N-2, N, \tag{4.4}$$

where, in order to make the notation compact, we have defined $\mathbf{v}^{(K)} \equiv \tilde{\mathbf{v}}^{(K)}/2 \equiv \mathbf{u}^{(K)}$. The coefficients C_l are determined as the solution of the system of algebraic equations $T_n(\rho_0) = 0, n = 1, 3 \ldots, N$ (the remaining boundary conditions at $-\rho_0$ are automatically fulfilled due to the symmetry of the interval $[A, B] = [-\rho_0, \rho_0]$). Explicitly the equations read

$$\sum_{j=1}^{K} C_j \left(v_n^{(j)} \exp(\tfrac{\lambda_j \rho_0}{2a\tau}) + u_n^{(j)} \exp(-\tfrac{\lambda_j \rho_0}{2a\tau})\right) = \frac{1}{2\Gamma}\left(\rho_0 v_n^{(K)} + 2a\tau \tilde{u}_n^{(K)}\right)^2, \quad n = 1, 3 \ldots, N. \tag{4.5}$$

Equations (4.4) and (4.5) are the main analytical results of the present paper. They reduce the problem of the first mean passage time from the finite interval in the case of the pure driving by the pre-Gaussian noise in the form of N independent telegraph processes to a solution of the system of linear algebraic equations (4.5). The results are valid for arbitrary values of the parameters Γ and τ characterizing the stochastic driving, the number N of independent driving processes, and the length of the interval $2\rho_0$.

The simplest case $K = N = 1$ (one telegraph noise) corresponds to the dichotomous Markov process [5] and gives

$$T(\rho) = T_{-1}(\rho) + T_1(\rho) = \frac{1}{2\Gamma}\left(\rho_0^2 - \rho^2\right) + \rho_0\sqrt{\frac{\tau}{\Gamma}}. \tag{4.6}$$

For reference let us remind the reader that the white noise limit (1.3) [$\tau \to 0$ in (1.2)] gives, under the assumption of absorbing boundaries at $\rho = \pm\rho_0$ [5, 20]:

$$T(\rho) = \frac{1}{2\Gamma}\left(\rho_0^2 - \rho^2\right). \tag{4.7}$$

The simplicity of the solution (4.4), (4.5) allows also an effective numerical investigation of the dependence of the solution on the number N. Before presenting in the next section results of this analysis let us make the following, interesting observation. Generally, the system is characterized by three independent parameters: the correlation time of the driving process τ, its strength Γ, and the length of the confining interval $2\rho_0$. It is, however, easy to prove that the dimensionless times T_n/τ as functions of the dimensionless coordinate $x = \rho/\rho_0$ depends on these parameters only through their dimensionless combination:

$$\gamma^2 \equiv \frac{\rho_0^2}{\Gamma\tau}. \tag{4.8}$$

Indeed, from (2.3) we have $a\tau = \sqrt{\Gamma\tau/N}$ and Eqs. (4.4) and (4.5) can be rewritten, respectively, as

$$\frac{T_n}{\tau} = \sum_{j=1}^{K} \frac{C_j}{\tau}\left[v_n^{(j)} \exp\left(\tfrac{\lambda_j\sqrt{N}\gamma}{2} x\right) + u_n^{(j)} \exp\left(-\tfrac{\lambda_j\sqrt{N}\gamma}{2} x\right)\right] - \frac{1}{2N}(\gamma x + n)^2,$$

$$n = -N, -N+2, \ldots, N-2, N, \tag{4.9}$$

and

$$\sum_{j=1}^{K} \frac{C_j}{\tau}\left[v_n^{(j)} \exp\left(\tfrac{\lambda_j\sqrt{N}\gamma}{2}\right) + u_n^{(j)} \exp\left(-\tfrac{\lambda_j\sqrt{N}\gamma}{2}\right)\right] = \frac{1}{2N}(\gamma + n)^2, \quad n = 1, 3 \ldots, N, \tag{4.10}$$

hence C_j/τ, as well as $T_n(x)/\tau$, depend, for fixed x, only on γ, and N,

$$T_n = \tau f_n\left(\frac{\rho_0}{\sqrt{\Gamma\tau}}, N\right). \tag{4.11}$$

In the limiting case $N \to \infty$ of the Ornstein-Uhlenbeck process (2.5) we obtain thus

$$T = \tau f\left(\frac{\rho_0}{\sqrt{\Gamma\tau}}\right), \tag{4.12}$$

where f is some universal function. The scaling law (4.12) seems to us remarkable and not transparent from the original formulation of the problem of first-passage time in the presence of the Ornstein-Uhlenbeck process.

Let us close this section with a remark, that straightforward summations involving binomial coefficients and the hypergeometric function lead to the following expression for MFPT for the pre-Gaussian noise (2.5)

$$\frac{T}{\tau} = \sum_{j=1}^{K} \frac{C_j}{\tau}\left(\frac{\eta_j + 1}{2}\right)^{K+j-1}\left(\frac{\eta_j - 1}{2\eta_j}\right)^{K-j} \times \cosh\left(\frac{\lambda_j\sqrt{N}}{2\gamma} x\right) - \frac{1}{2N}\left[(\gamma x)^2 + N\right]. \tag{4.13}$$

V. RESULTS FOR LARGE N

Figure 1 shows the mean first-passage time $T(\rho)$ Eq. (2.5) calculated from (4.4) and (4.5) for $\gamma = 1$ and various values of N. It is clear that the solutions converge rather rapidly for $N > 20$ to a universal curve. The curves for $N = 23$ and $N = 25$ are not distinguishable for all practical purposes and we can assume that $N = 25$ approximate reasonably the case $N \to \infty$, i.e., the Ornstein-Uhlenbeck process. This observation is fairly independent on the actual values of the parameters τ, Γ, and ρ_0 as indicated by Figs. 2 and 3 in which γ^2 differs by an order from the one of Fig. 1. The effectiveness of the approximation of the Ornstein-Uhlenbeck process by a relative few composed independent random telegraph signals was already observed in other models [19].

Using the above approximation (i.e., $N = 25$ as corresponding to the limiting case of the Ornstein-Uhlenbeck process) we were able to investigate the dependence of the mean first-passage time T on the correlation time of the noise τ and its intensity Γ. Figure 4 shows the dependence of the mean escape time from the middle ($\rho = 0$) of an interval of the length $2\rho_0 = 2$ for different values of the intensity of the noise Γ. The dependence of T on Γ is

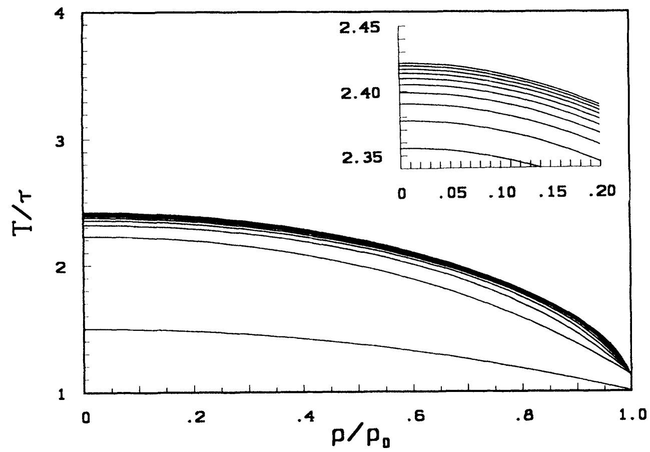

FIG. 1. Dimensionless total first mean passage time T/τ as a function of the dimensionless coordinate ρ/ρ_0 for $\gamma^2 = 1$ and $N = 1, 3, \ldots, 25$. Inset: detail in the vicinity of $\rho = 0$. The consecutive curves corresponding to a growing number of independent random telegraph signals N converge rapidly to a universal curve.

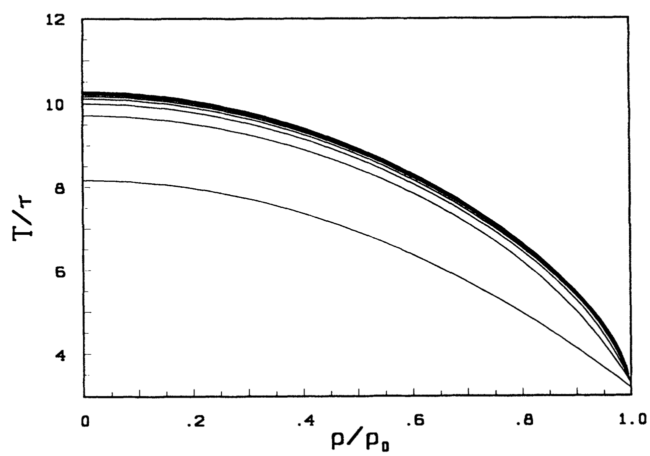

FIG. 2. Same as Fig. 1 but for $\gamma^2 = 10$.

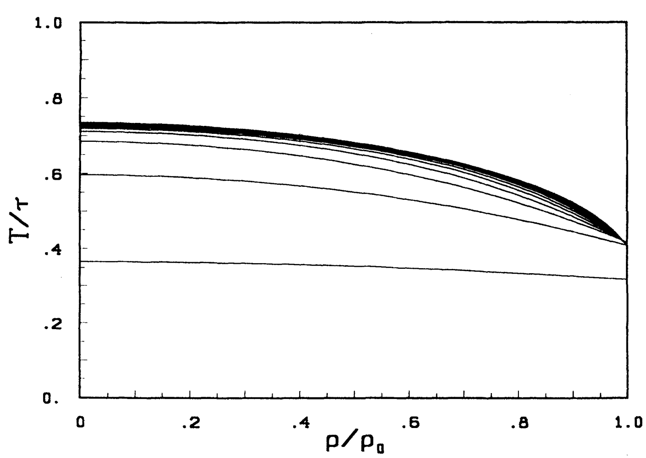

FIG. 3. Same as Fig. 1 but for $\gamma^2 = 0.1$

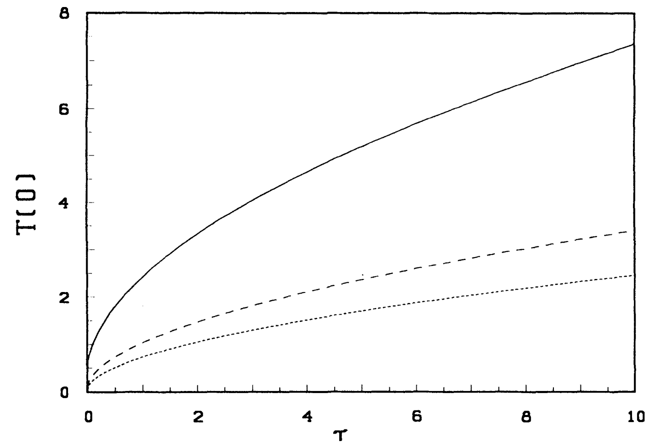

FIG. 4. Mean first escape from the middle of the interval $[-\rho_0, \rho_0]$ time for $N = 25$ as a function of the correlation time τ for different values of the noise intensity Γ: $\Gamma = 1$ (solid line), $\Gamma = 5$ (dashed line), and $\Gamma = 10$ (dotted line).

presented in Fig. 5, which corresponds to the same situation as in Fig. 4, i.e., the escape time from the middle of the interval $[-1, 1]$ for different values of the correlation time τ.

Numerical simulations of MFPT for the Ornstein-Uhlenbeck process were performed in Ref. [12], where results corresponding to our $T(0)$ for $\rho_0 = 1$, $\Gamma = 0.1$, and several values of τ were reported. Figure 6 shows the dependence of $T(0)$ on the correlation time τ for the above values of parameters. The dotted line corresponds to the interpolation formula suggested in Ref. [12] and given by

$$T(\rho) = \frac{\rho_0^2 - \rho^2}{2\Gamma} + 1.4\sqrt{\frac{\tau}{\Gamma}}. \tag{5.1}$$

The curve corresponding to the dichotomous noise ($N =$

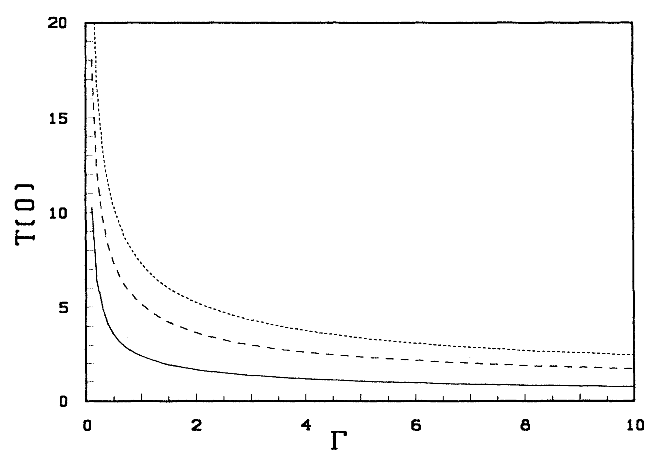

FIG. 5. Mean first escape from the middle of the interval $[-\rho_0, \rho_0]$ time for $N = 25$ as a function of the noise intensity Γ for different values of the correlation time τ: $\tau = 1$ (solid line), $\tau = 5$ (dashed line), and $\tau = 10$ (dotted line).

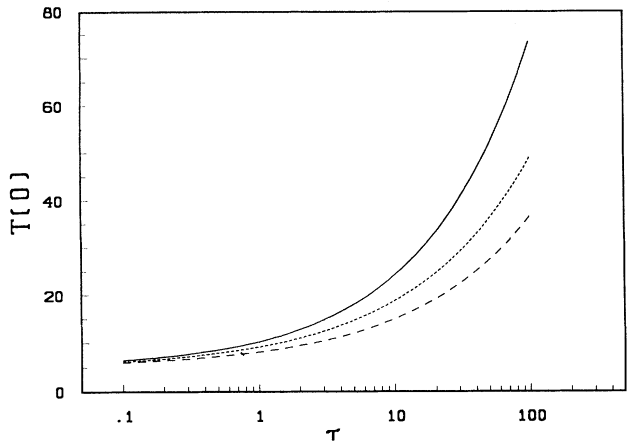

FIG. 6. Dependence of the escape time $T(0)$ from the interval $[-1, 1]$ on the correlation time τ. The noise intensity is $\Gamma = 0.1$. Our results (solid line), interpolation formula of Ref. [12] (dotted line), dichotomous (single telegraph, $N = 1$) noise (dashed line).

1) is given for reference. Although the qualitative behavior of $T(0)$ as a function of τ is very similar to the one reported in Ref. [12] there is a significant quantitative discrepancy between our results and those of Ref. [12]. The discrepancy grows with the correlation time τ.

VI. SUMMARY

In the present paper we have investigated the problem of the mean first-passage time (escape time) for a colored Wiener-Lévy stochastic process driven by N random telegraph signals or the Ornstein-Uhlenbeck noise. We have reduced the case of the Wiener-Lévy process driven by N random telegraph signals to a solution of a system of linear algebraic equations. The Wiener-Lévy stochastic process driven by the Ornstein-Uhlenbeck noise is recovered in the limiting case of taking the number of independent telegraph signals tending to infinity. From the derived equations we have deduced exact scaling laws [Eqs. (4.11) and (4.12)] obeyed by the mean first-passage times in the cases of RTS and Ornstein-Uhlenbeck stochastic driving. The scaling law involves the length of the driving interval as well as the intensity and the correlation time of the process. These parameters are combined to a single constant, i.e., a single parameter on which the dimensionless MFPT depends. The rapid convergence of the random telegraph results when their number N grows (cf. Figs. 1, 2, and 3) allows for effective calculations concerning the Ornstein-Uhlenbeck process. Using the finite-N approximations we presented the dependence of the first-passage time T for the Ornstein-Uhlenbeck process on the intensity and correlation time of the process.

ACKNOWLEDGEMENT

The present work was carried out under the project funded by Grant No. 2 0001 91 01 of Polish State Committee for Scientific Research.

* Present address: Fachbereich Physik, Unversität-GHS Essen, 4300 Essen, Federal Republic of Germany. Permanent address: Center for Theoretical Physics, Polish Academy of Sciences, aleja Lotników 32/46, 02-668 Warsaw, Poland.

† Also at Center for Advanced Studies, University of New Mexico, Albuquerque, NM 87131. Permanent address: Institute of Theoretical Physics, Warsaw University, Hoża 69, 00-681 Warsaw, Poland.

[1] P. Hänggi and P. Talkner, Phys. Rev. Lett. **51**, 2242 (1983).
[2] P. Hänggi and P. Talkner, Phys. Rev. A **32**, 1934 (1985).
[3] V. Balakrishnan, C. Van den Broeck, and P. Hänggi, Phys. Rev. A **38**, 2242 (1988).
[4] M. A. Rodriguez and L. Pesquera, Phys. Rev. A **34**, 4532 (1986).
[5] J. Masoliver, K. Lindenberg, and B. J. West, Phys. Rev. A **33**, 2177 (1986); **34**, 1481 (1986).
[6] K. Lindenberg, B. J. West, and G. P. Tsironis, Rev. Solid State Sci. **3**, 143 (1989).
[7] G. H. Weiss, J. Masoliver, K. Lindenberg, and B. J. West, Phys. Rev. A **36**, 1435 (1989).
[8] R. F. Fox, Phys. Rev. A **33**, 467 (1986); **34**, 4525 (1986).
[9] R. F. Fox, Phys. Rev. A **37**, 911 (1988).
[10] G. P. Tsironis and P. Grigolini, Phys. Rev. A **38**, 3749 (1988).
[11] L. Ramirez-Piscina, J. M. Sancho, F. J. de la Rubia, K. Lindenberg, and G. P. Tsironis, Phys. Rev. A **40**, 2120 (1989).
[12] V. Palleschi, Phys. Lett. A **128**, 318 (1988).
[13] R. Manella and V. Palleschi, Phys. Rev. A **40**, 3381 (1989).
[14] J. Müller and J. Schnakenberg, Ann. Phys. (Leipzig) (to be published).
[15] J. L. Doob, Ann. Math. **43**, 351 (1942).
[16] G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. **36**, 823 (1930).
[17] M. Kuś, E. Wajnryb, and K. Wódkiewicz, Phys. Rev. A **42**, 7500 (1990).
[18] M. Kuś, E. Wajnryb, and K. Wódkiewicz, Phys. Rev. A **43**, 4167 (1991); **44**, 4080 (1991).
[19] K. Wódkiewicz, J. H. Eberly, and B. W. Shore, J. Opt. Soc. Am. B **1**, 398 (1984); Phys. Rev. A **30**, 2381 (1984).
[20] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1983).
[21] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (National Bureau of Standards, Washington, DC, 1964).